$$x^{1.5}_{1,0} = 7.22215737 \times 10^{-2} \qquad x^{1.5}_{2,0} = 1.03747215 \times 10^{-1}$$

$$u^{0.5}_{1,0} = 3.93469340 \times 10^{-1}$$

$$u^{1.0}_{1,0} = 6.32120563 \times 10^{-1}$$

$$u^{1.5}_{1,0} = 7.76869840 \times 10^{-1}$$

where, again, we have chosen $k=0$. Hence,

$$[\hat{\Phi}] = \begin{bmatrix} 4.91609053 \times 10^{-2} & 6.97934045 \times 10^{-2} & 7.22215737 \times 10^{-2} \\ 2.24728763 \times 10^{-2} & 1.14113749 \times 10^{-1} & 1.03747215 \times 10^{-1} \\ 3.93469340 \times 10^{-1} & 6.32120563 \times 10^{-1} & 7.76869840 \times 10^{-1} \end{bmatrix}$$

$$[\hat{\Psi}] = \begin{bmatrix} 7.16337814 \times 10^{-2} & 1.83907152 \times 10^{-1} & 1.75968791 \times 10^{-1} \\ -9.58924274 \times 10^{-1} & -5.44021110 \times 10^{-1} & 6.50287840 \times 10^{-1} \\ 4.91609053 \times 10^{-2} & 6.97934045 \times 10^{-2} & 7.22215737 \times 10^{-2} \\ 2.24728763 \times 10^{-2} & 1.14113749 \times 10^{-1} & 1.03747215 \times 10^{-1} \end{bmatrix}$$

$$[C] = \begin{bmatrix} -1.81247486 \times 10^{-7} & 9.99999969 \times 10^{-1} & 2.38484290 \times 10^{-8} \\ -9.99999800 \times 10^{1} & -2.20344477 \times 10^{-6} & 9.99999731 \end{bmatrix}$$

where, again, for comparison, the data belong to

$$[C] = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 10 \end{bmatrix}$$

Conclusion

The method of Poisson moment functionals has been extended to include synthesis of the state equations when internal and/or external forces are prescribed. The method is advantageous as process signals are exponentially weighted and integrated, thus minimizing some of the effects of zero mean additive noise.

Although the method does require numerical convolution of each process signal (input and output), or, alternately, numerical forward and inverse Laplace transformation of each process signal in order to implement the filter chain, the actual solution for the unknown coefficients involves inversion of a matrix of rank $(n+m)$. However, when more sensors are available than is necessary to determine the unknown $[C]$ matrix, the algorithm is easily modified. The matrices $[\Phi]$ and $[\Psi]$ (or $[\hat{\Phi}]$ and $[\hat{\Psi}]$) become rectangular, with more rows than columns. Solution is then obtained by the least squares formulation,

$$[C] = [Y][\Phi]^T[\Psi]([\Phi]^T[\Phi])^{-1} \tag{14}$$

Finally, PMF provides a means of identification in continuous time, eliminating the necessity of transformation from discrete to continuous time of identified parameters.

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Non-Fourier Thermal Stresses in a Circular Disk

M. H. Sadd*

University of Rhode Island, Kingston, Rhode Island

and

C. Y. Cha†

Pennsylvania State University, Middletown, Pennsylvania

Introduction

OVER the past few years, research has been conducted dealing with departures from the classical Fourier conduction law. The reason for this research was to eliminate the paradox of an infinite thermal wave speed and thus provide a theory to explain the experimental data on "second sound" in materials such as liquid and solid helium at low temperatures.[1,2] In addition to low-temperature applications, non-Fourier theories may also be useful for high heat flux, short time behavior as found, for example, in laser material interaction.

Fourier conduction theory relates the heat flux vector $\boldsymbol{q}$ to the temperature gradient ∇T, by the relation

$$q = -k \nabla T \tag{1}$$

where the constant k is the thermal conductivity. Only isotropic and homogeneous media will be considered herein. Equation (1), along with the conservation of energy, gives the classical parabolic heat equation

$$\alpha \nabla^2 T = \frac{\partial T}{\partial t} \tag{2}$$

where $\alpha = k/\rho c^*$ = thermal diffusivity, ρ = mass density, and c^* = specific heat capacity. Relation (2) produces temperature solutions that correspond to an infinite speed of heat propagation.

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*Associate Professor, Department of Mechanical Engineering and Applied Mechanics.

†Assistant Professor, Mechanical Engineering Design Department.

To eliminate this shortcoming of Fourier theory, Cattaneo[3,4] and Vernotte[5,6] proposed a damped wave model for heat conduction of the form:

$$q = -k\nabla T - \tau\frac{\partial q}{\partial t} \tag{3}$$

The material constant τ is called the thermal relaxation time and physically represents the result of a finite thermal communication time between material points. Equation (3), combined with conservation of energy, gives the hyperbolic heat conduction equation

$$\alpha\nabla^2 T = \frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2} \tag{4}$$

Equation (4) can be recognized as the telegraph equation that yield solutions of the form of temperature waves propagating with speed $c=\sqrt{\alpha/\tau}$ while decaying exponentially with time.

With regard to heat conduction, many previous research efforts have been devoted to finding solutions to the non-Fourier heat equation. The authors[7] have recently developed such a solution for axisymmetric cylindrical domains. Popov[8] and Lord and Shulman[9] were the first researchers to include this non-Fourier conduction theory within the framework of a thermoelasticity theory. Other research dealing with non-Fourier thermoelasticity are given by Achenbach,[10] Norwood and Warren,[11] Nayfeh,[12-14] Kosinski and Szmit,[15] Tokuoka,[16] Kao,[17] and Ignaczak.[18]

Since most of this previous thermoelastic work has been for one-dimensional Cartesian geometries only, the purpose of this Note is to present a non-Fourier thermoelastic study in a cylindrical geometry. Specifically, the present study is concerned with the axisymmetic, uncoupled thermoelastic problem in the interior region, $r\leq a^*$. The displacements and stresses are thermally generated by a dynamic temperature boundary condition.

Formulation

Within the context of linear thermoelasticity theory for axisymmetric cylindrical geometries, the energy, non-Fourier thermal conduction, and momentum equations are

$$\rho c^*\frac{\partial T}{\partial t} + (3\lambda+2\mu)\bar{\alpha}T_0\left(\frac{\partial^2 u}{\partial r\partial t} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = -\frac{\partial q}{\partial r} - \frac{q}{r} \tag{5}$$

$$\frac{\alpha}{c^2}\frac{\partial q}{\partial t} + q = -k\frac{\partial T}{\partial r} \tag{6}$$

$$\frac{1}{c_e^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{(3\lambda+2\mu)\bar{\alpha}}{(\lambda+2\mu)}\frac{\partial T}{\partial r} \tag{7}$$

where c_e is the mechanical stress wave speed given by $c_e = \sqrt{(\lambda+2\mu)/\rho}$, λ and μ the traditional elastic constants, $\bar{\alpha}$ the coefficient of thermal expansion, and T_0 the reference temperature. The radial heat flux q, and displacement u, along with the temperature, are assumed to be functions of the radial coordinate and time.

The problem to be considered consists of the region interior to the circle $r=a^*$ with step function temperature and clamped boundary conditions. It will be convenient to introduce the following dimensionless variables

$$\delta = \frac{r}{2\sqrt{\alpha\tau}}, \quad \beta = \frac{t}{2\tau}, \quad \theta = \frac{T-T_i}{T_s-T_i}$$

$$u' = \frac{u}{A}, \quad A = \frac{2(3\lambda+2\mu)\alpha\bar{\alpha}}{(\lambda+2\mu)c}(T_s-T_i) \tag{8}$$

where T_i is the initial uniform medium temperature and T_s the constant surface temperature to be applied at the boundary $r=a^*$.

In terms of the dimensionless variables, the problem thus may be written as

$$\frac{\partial^2\theta}{\partial\delta^2} + \frac{1}{\delta}\frac{\partial\theta}{\partial\delta} = \frac{\partial^2\theta}{\partial\beta^2} + 2\frac{\partial\theta}{\partial\beta} \tag{9}$$

$$\frac{\partial^2 u'}{\partial\delta^2} + \frac{1}{\delta}\frac{\partial u'}{\partial\delta} - \frac{1}{\delta^2}u' - \left(\frac{c}{c_e}\right)^2\frac{\partial^2 u'}{\partial\beta^2} = \frac{\partial\theta}{\partial\delta} \tag{10}$$

where we have dropped the coupling term to arrive at Eq. (9). The boundary and initial conditions become

$$\theta(\delta,0) = \frac{\partial\theta}{\partial\beta}(\delta,0) = 0$$

$$u(\delta,0) = \frac{\partial u}{\partial\beta}(\delta,0) = 0$$

$$\theta(a,\beta) = 1, \quad u(a,\beta) = 0 \tag{11}$$

where $a = a^*/2\sqrt{\alpha\tau}$.

Solution and Results

Equations (9) and (10) can be solved by using the method of Laplace transforms. Details of the solution procedure are straightforward but lengthy, and will not be given here. Details can be found in Refs. 7 and 19.

Since the results for the temperature distribution have been given elsewhere,[7] we present herein only the displacment solution. This solution reads

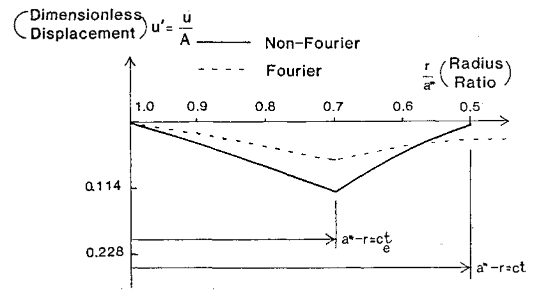

Fig. 1 Comparison of Fourier and non-Fourier displacement profiles for the interior problem ($\beta=0.5$), $c=1.6c_e$.

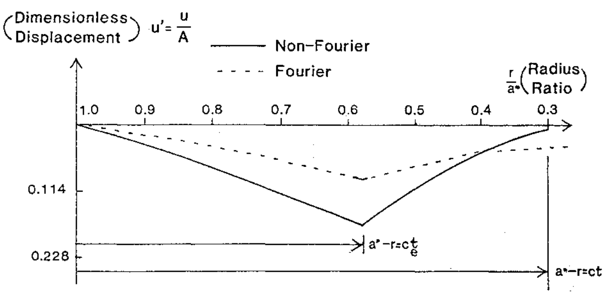

Fig. 2 Comparison of Fourier and non-Fourier displacement profiles for the interior problem ($\beta=0.7$) $c=1.6c_e$.

$$
\begin{aligned}
u'(\delta,\beta) = {} & \sum_{n=1}^{\infty} \frac{J_1(\lambda_n\delta)}{J_1(\lambda_n a)} [(A_n \sin k_n\beta + B_n \cos k_n\beta) e^{-\beta} - P_n e^{-B\beta} + C_n] + \frac{\bar{\delta}}{\bar{a}} \sum_{k=1}^{\infty} \Big\{F_n V_n [e^{-\beta}(\sin k_n\beta + k_n \cos k_n\beta) - k_n] \\
& + H_n V_n [e^{-\beta}(\cos k_n\beta - k_n \sin k_n\beta) - 1] + H_n \left[\frac{1}{B}(1 - e^{-B\beta})\right]\Big\} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_m J_1(\lambda_m\bar{\delta}) \{F_n[\cos\lambda_m\beta\{e^{-\beta}[-E_{mn}\sin(k_n-\lambda_m)\beta \\
& - (k_n-\lambda_m)E_{mn}\cos(k_n-\lambda_m)\beta - G_{mn}\sin(k_n+\lambda_m)\beta - (k_n+\lambda_m)G_{mn}\cos(k_n+\lambda_m)\beta] + (k_n-\lambda_m)E_{mn} \\
& + (k_n+\lambda_m)G_{mn}\} + \sin\lambda_m\beta\{e^{-\beta}[(k_n-\lambda_m)E_{mn}\sin(k_n-\lambda_m)\beta - E_{mn}\cos(k_n-\lambda_m)\beta - (k_n+\lambda_m)G_{mn}\sin(k_n+\lambda_m)\beta \\
& + G_{mn}\cos(k_n+\lambda_m)\beta] + E_{mn} - G_{mn}\}] + H_n[\cos\lambda_m\beta\{e^{-\beta}[(k_n-\lambda_m)E_{mn}\sin(k_n-\lambda_m)\beta - E_{mn}\cos(k_n-\lambda_m)\beta \\
& + (k_n+\lambda_m)G_{mn}\sin(k_n+\lambda_m)\beta - G_{mn}\cos(k_n+\lambda_m)\beta)] + E_{mn} + G_{mn}\} + \sin\lambda_m\beta\{e^{-\beta}[E_{mn}\sin(k_n-\lambda_m)\beta \\
& + (k_n-\lambda_m)E_{mn}\cos(k_n-\lambda_m)\beta - G_{mn}\sin(k_n+\lambda_m)\beta - (k_n+\lambda_m)G_{mn}\cos(k_n+\lambda_m)\beta] - (k_n-\lambda_m)E_{mn} + (k_n+\lambda_m)G_{mn}\}] \\
& - H_n[-R_m\cos\lambda_m\beta + S_m\sin\lambda_m\beta + e^{-B\beta}R_m]\}
\end{aligned}
\tag{12}
$$

where

$$
A_n = \frac{1}{ak_n}\left\{\frac{(B-2)(B-1)}{((B-1)^2+k_n^2)} - \frac{2}{(1+k_n^2)}\right\}, \quad B_n = \frac{1}{a}\left\{\frac{(2-B)}{((B-1)^2+k_n^2)} - \frac{2}{(1+k_n^2)}\right\}, \quad P_n = \frac{(2-B)}{a((B-1)^2+k_n^2)}
$$

$$
C_n = \frac{2}{a(1+k_n^2)}, \quad F_n = \frac{B}{ak_n}\left\{1 - \frac{(B-1)(B-2)}{(B-1)^2+k_n^2}\right\}, \quad H_n = \frac{B}{a}\left\{\frac{(B-2)}{(B-1)^2+k_n^2}\right\}, \quad V_n = \frac{1}{1+k_n^2}
$$

$$
E_{mn} = \frac{1}{2[1+(k_n-\lambda_m)^2]}, G_{mn} = \frac{1}{2[1+(k_n+\lambda_m)^2]}, R_m = -\frac{B}{B^2+\lambda_m^2}, \quad S_m = \frac{\lambda_m}{B^2+\lambda_m^2}, D_m = \frac{2}{\lambda_m \bar{a} J_0(\lambda_m \bar{a})}, \quad B = \frac{2}{1-(c/c_e)^2}
$$

where λ_n are the roots of $J_0(\lambda a) = 0$, λ_m are the roots of $J_1(\lambda\bar{a}) = 0$, $k_n^2 = \lambda_n^2 - 1$, and $\bar{a} = ca/c_e$ and $\bar{\delta} = c\delta/c_e$. The stresses can be computed from the previous displacement solution using Hooke's law; such results have been given by Cha.[19]

Displacement profiles of this anlaysis are shown in Figs. 1 and 2 for different values of dimensionless time. For comparison purposes, the corresponding Fourier results are also shown. With regard to the non-Fourier theory, two wave fronts exist: a mechanical wave front is located at $a^* - r = c_e t$, while a thermal wave front is found at $a^* - r = ct$. Note that these results are for the case $c = 1.6\ c_e$. No distinct thermal wave front is predicted by the classical Fourier theory. It is apparent from these figures that non-Fourier displacements will be larger (in absolute value) than those from Fourier theory. Non-Fourier stress magnitudes are also higher than the classical results. Since the relaxation time $\tau \cong 10^{-9}$ s, differences between the two theories disappear rapidly as time progress, and it is only for very short times that non-Fourier effects will be significant. However, considering laser-material interaction problems with pulse times in the nanosecond range, it is believed that a non-Fourier modeling approach would be more correct in predicting such material behavior.

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